

An xp model on AdS_2 spacetime

Javier Molina-Vilaplana⁽¹⁾ and Germán Sierra⁽²⁾

⁽¹⁾ Department of Systems Engineering and Automation, Technical University of Cartagena, Cartagena, Spain

⁽²⁾ Instituto de Física Teórica, UAM-CSIC, Madrid, Spain

E-mail: javi.molina@upct.es, german.sierra@uam.es

Abstract. In this paper we formulate the xp model on the AdS_2 spacetime. We find that the spectrum of the Hamiltonian has positive and negative eigenvalues, equal in magnitude, given by a harmonic oscillator with a zero point energy parameterized by the AdS radius, measured in units of a fundamental length of the model. We also construct the generators of the isometry group $SO(2,1)$ of the AdS_2 spacetime, and discuss the relation with conformal quantum mechanics.

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1. Introduction

In 1999 Berry, Keating and Connes suggested that a spectral realization of the Riemann zeros could be achieved by quantizing the classical Hamiltonian $H = xp$, where x and p are the position and momenta of a particle moving in one dimension [1, 2]. That realization would provide a proof of the celebrated Riemann hypothesis [3] along the lines of the old Hilbert and Polya conjecture, according to which the Riemann zeros are oscillation frequencies of some physical system [4, 5]. Several recent works have been devoted to clarify the possible relation of the xp model with the Riemann zeros [6]-[12]. In references [7, 8, 11] it was advocated that one needs to modify the xp Hamiltonian in order to have a discrete spectrum, since the standard quantization of xp yields a continuum [13, 14]. The new Hamiltonians take the form $H = w(x)(p + \ell_p^2/p)$, where $w(x) = x$ ($x \geq \ell_x$) and $w(x) = x + \ell_x^2/x$ ($x \leq 0$), where $\ell_{x,p}$ have dimensions of length and momenta and such that their product is the Planck's constant, i.e. $\ell_x \ell_p = 2\pi\hbar$ [7, 8, 11]. Quite interestingly, the spectra of the corresponding quantized Hamiltonians, E_n , agrees asymptotically with the average part of the Riemann-Mangoldt counting formula, $\langle n(t) \rangle = \frac{t}{2\pi} \log \frac{t}{2\pi e}$, with the identification $t_n = E_n/\hbar$ [3]. Unfortunately, the fluctuation part of the Riemann-Mangoldt formula, given by $n_{\text{fl}}(t) = \frac{1}{\pi} \text{Im} \zeta(\frac{1}{2} + it)$, is not reproduced in the spectrum which is completely smooth.

The previous results suggest that further modifications of the xp model are required to make full contact with the Riemann zeros. This idea motivated the study of the general Hamiltonian $H = w(x)(p + \ell_p^2/p)$ for arbitrary functions $w(x)$ [11]. It turned out that these Hamiltonians describe the motion of a relativistic particle moving in a 1+1 spacetime whose metric is determined by $w(x)$. The Riemann scalar curvature, given by $\mathcal{R}(x) = -2w''(x)/w(x)$, vanishes identically for the lineal potential $w(x) = x$, and asymptotically for $w(x) = x + \ell_x^2/x$. This result points towards an intriguing connection between the smooth Riemann zeros and asymptotically flat 1+1 spacetimes. In reference [11] it was also found that the function $w(x) = w_0 \cosh(x/R)$ yields a spacetime with constant negative curvature $\mathcal{R} = -2/R^2$, which corresponds to an anti-de-Sitter space (AdS_2), with radius R . For this reason we shall denote this model as xp - AdS_2 . Moreover, the semiclassical spectrum of the xp - AdS Hamiltonian is given by time conjugated pairs $(E_n, -E_n)$, with $E_n = w_0(n + R\ell_p/\hbar + 1/2)$, $n = 0, 1, \dots, \infty$. The positive branch of the spectrum is hence that of a harmonic oscillator, whose zero point energy is related to the product of the radius R and the elementary momenta ℓ_p entering into the Hamiltonian.

The first aim of this paper is to show that this semiclassical spectrum is exact in the quantum theory. In this manner we find one rare example where semiclassical results are exact. The isometry group of the AdS_2 metric is $SO(2, 1)$. This Lie group is also the conformal group in $0 + 1$ dimensions and is therefore the symmetry group of conformal quantum mechanics [15, 16, 17, 18]. This suggests a connection between gravity on AdS_2 and a conformal quantum mechanics living on the one-dimensional boundary of AdS_2 . However, to find explicit realizations of this correspondence has shown to be rather

elusive, motivating several works to clarify the AdS_2/CFT_1 correspondence [19]–[24]. In string theory, the latter duality is also relevant because the AdS_2 geometry is the factor appearing in the near horizon geometry of extremal black holes in any dimension [25]. In [26] authors considered the dynamics of a (super) particle in the near horizon geometry of an extremal Reissner-Nordström black hole which coincides with the $AdS_2 \times S_2$ geometry. In this region, the dynamics of the particle is governed by the model of conformal mechanics described by the Hamiltonian of [16]. In the context of the black hole, the $SO(2, 1)$ symmetry of this model is an inherited feature of the $SO(2, 1)$ isometries of AdS_2 .

In this paper we also investigate the $SO(2, 1)$ symmetry of the xp - AdS_2 model and its connection with the some of the previous works.

2. The xp - AdS_2 model

The general xp Hamiltonians are defined by [11]

$$H = w(x) \left(p + \frac{\ell_p^2}{p} \right) \quad (1)$$

where x and p are the position and momentum of a particle moving in an interval D of the real line, ℓ_p is a parameter with dimensions of momentum, and $w(x) > 0$ is a positive function with dimensions of velocity. We shall take D to be the whole line, but other choices, as the half-line, are also possible. The role of the term proportional to $1/p$ is to forbid the particle to scape to infinity, so that the spectrum could be discrete depending on the function of $w(x)$ [7]. The Hamiltonian (1) breaks time reversal symmetry, under which $H \rightarrow -H$. Moreover, the sign of p is a conserved quantity since $w(x) > 0$. We shall restrict ourselves to the choice $\text{sign } p > 0$.

The action associated to (1) is the one of a relativistic massive particle

$$S = \int dt L = -\ell_p \int \sqrt{-ds^2}, \quad (2)$$

moving in a spacetime with metric

$$\frac{1}{4}ds^2 = -w(x)^2 dt^2 + w(x) dt dx. \quad (3)$$

The momenta ℓ_p plays the role of mc , where m is a mass and c the speed of light. The special form of the metric (3) means that t and x are null coordinates. The Riemann scalar curvature is

$$\mathcal{R}(x) = -2 \frac{\partial_x^2 w(x)}{w(x)}, \quad (4)$$

and it is constant and negative for the choice

$$w(x) = w_0 \cosh \frac{x}{R} \rightarrow \mathcal{R}(x) = -\frac{2}{R^2}, \quad x \in (-\infty, \infty). \quad (5)$$

This fact implies that the particle follows the geodesics of the AdS_2 spacetime (see Appendix B). The metric (3) becomes

$$\frac{1}{4}ds^2 = -w_0^2 \cosh^2\left(\frac{x}{R}\right) dt^2 + w_0 \cosh\left(\frac{x}{R}\right) dt dx. \quad (6)$$

The change of variables

$$\sinh \frac{x}{R} = \tan \theta, \quad t = \frac{R}{2w_0}(\tau + \theta), \quad \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \quad \tau \in (-\infty, \infty) \quad (7)$$

brings (6) into the standard form of the AdS_2 metric (see Appendix A)

$$ds^2 = \frac{R^2}{\cos^2 \theta} (-d\tau^2 + d\theta^2). \quad (8)$$

3. Semiclassical spectrum

The number of semiclassical energy levels, $n(E)$, between 0 and $E > 0$, is given by [11]

$$n(E) + \frac{1}{2} = \frac{1}{2\pi\hbar} \int_{-x_M}^{x_M} \frac{dx}{w(x)} \sqrt{E^2 - 4\ell_p^2 w^2(x)} \quad (9)$$

where we have assumed that $w(x)$ is an even function, as in eq. (5). x_M is the turning point of the classical trajectories, i.e. $E = 2w(x_M)$. The constant $\frac{1}{2}$ has been included to account for the Maslow phase associated to the classical trajectories (see Appendix B). Using the variable θ (eq.(7)), eq.(9) becomes

$$n + \frac{1}{2} = \frac{2R\ell_p}{\pi\hbar} \int_0^{\theta_M} d\theta \sqrt{\varepsilon^2 - \frac{1}{\cos^2 \theta}} = \frac{R\ell_p}{\hbar} (\varepsilon - 1), \quad (10)$$

where

$$\varepsilon = \frac{E}{2w_0\ell_p} = \frac{1}{\cos \theta_M}. \quad (11)$$

Hence the semiclassical spectrum is

$$E_n = \frac{2\hbar w_0}{R} \left(n + \frac{R\ell_p}{\hbar} + \frac{1}{2} \right), \quad n = 0, 1, \dots \quad (12)$$

which coincides with the harmonic oscillator spectrum with a zero point energy that depends on the dimensionless constant

$$\kappa = \frac{R\ell_p}{\hbar}, \quad (13)$$

that may take any positive value.

4. Quantum spectrum of the xp -AdS model

The classical Hamiltonian (1) can be quantized in terms of the following normal ordered operator [11]

$$\hat{H} = u(x) \left(\hat{p} + \frac{\ell_p^2}{\hat{p}} \right) u(x), \quad (14)$$

where

$$u(x) = \sqrt{w(x)}. \quad (15)$$

The action of \hat{H} on a wave function is

$$(\hat{H}\psi)(x) = -i\hbar u(x) \frac{d}{dx} \{u(x)\psi(x)\} - i \frac{\ell_p^2}{\hbar} \int_x^\infty dy u(x) u(y) \psi(y), \quad (16)$$

and it is a symmetric operator, i.e.

$$\langle \psi_1 | \hat{H} \psi_2 \rangle = \langle \hat{H} \psi_1 | \psi_2 \rangle = 0, \quad (17)$$

provided the wave functions satisfies the conditions

$$\lim_{x \rightarrow \pm\infty} u(x)\psi(x) = 0, \quad \int_{-\infty}^\infty dx u(x) \psi(x) = 0. \quad (18)$$

The Schroedinger equation is

$$-i\hbar u(x) \frac{d}{dx} \{u(x)\psi(x)\} - i \frac{\ell_p^2}{\hbar} \int_x^\infty dy u(x) u(y) \psi(y) = E\psi(x), \quad (19)$$

and similarly

$$\hbar^2 \frac{d}{dx} \{u(x)\psi(x)\} + \ell_p^2 \int_x^\infty dy u(y) \psi(y) - iE\hbar \frac{\psi(x)}{u(x)} = 0. \quad (20)$$

Defining

$$\phi(x) = u(x)\psi(x), \quad (21)$$

the eq.(20) reads

$$\hbar^2 \frac{d\phi(x)}{dx} + \ell_p^2 \int_x^\infty dy \phi(y) - iE\hbar \frac{\phi(x)}{w(x)} = 0, \quad (22)$$

and the conditions (18)

$$\lim_{x \rightarrow \pm\infty} \phi(x) = 0, \quad \int_{-\infty}^\infty dx \phi(x) = 0. \quad (23)$$

To solve eq. (22) one takes a derivative obtaining

$$\hbar^2 \frac{d^2\phi(x)}{dx^2} - iE\hbar \frac{d}{dx} \left(\frac{\phi(x)}{w(x)} \right) - \ell_p^2 \phi(x) = 0. \quad (24)$$

Equations (24) and (23) imply (22) for functions $\phi(x)$ which decay sufficiently fast at $\pm\infty$. For the function

$$w(x) = w_0 \cosh \frac{x}{R}, \quad (25)$$

one gets

$$\partial_x^2 \phi(x) - \frac{iE}{\hbar w_0} \frac{1}{\cosh(x/R)} \partial_x \phi(x) + \left(\frac{iE}{\hbar w_0 R} \frac{\sinh(x/R)}{\cosh^2(x/R)} - \frac{\ell_p^2}{\hbar^2} \right) \phi(x) = 0. \quad (26)$$

In the limit $|x| \rightarrow \infty$ this eq. yields

$$\partial_x^2 \phi(x) - \frac{\ell_p^2}{\hbar^2} \phi(x) \sim 0, \quad (27)$$

so that the wave function decays asymptotically as

$$\phi(x) \rightarrow e^{-|x|\ell_p/\hbar}, \quad |x| \rightarrow \infty \quad (28)$$

To solve eq.(26) we first write it in terms of the angle variable θ ,

$$\cos^2 \theta \partial_\theta^2 \phi - \cos \theta (\sin \theta + i\alpha \cos \theta) \partial_\theta \phi + (i\alpha \sin \theta \cos \theta - \kappa^2) \phi = 0 \quad (29)$$

where

$$\alpha = \frac{ER}{\hbar w_0}. \quad (30)$$

Notice that the semiclassical result (12) implies for α

$$\alpha_n^{(\text{sc})} = 2n + 2\kappa + 1, \quad n = 0, 1, \dots \quad (31)$$

together with the negative values $-\alpha_n^{(\text{sc})}$. Next we define the function $f(\theta)$ as

$$f(\theta) = 2 \cos \theta \phi(\theta), \quad (32)$$

which satisfies

$$\cos^2 \theta \partial_\theta^2 f + \cos \theta (\sin \theta - i\alpha \cos \theta) \partial_\theta f + (1 - \kappa^2) f = 0. \quad (33)$$

Defining the complex variable

$$z = e^{2i\theta}, \quad (34)$$

eq. (33) becomes

$$z(z+1)^2 \partial_z^2 f + \frac{1}{2}(z+1)[(1-\alpha)z+3-\alpha] \partial_z f + (\kappa^2 - 1)f = 0, \quad (35)$$

One can verify that if $f(z)$ is a solution of this equation then $f(z^{-1})$ is a solution with α replaced by $-\alpha$. In this manner one gets the negative energy solutions from the positive ones. The conditions (23) read

$$\lim_{z \rightarrow e^{\pm i\pi}} \frac{f(z)}{(1+z)} = 0, \quad (36)$$

$$\int_{\mathcal{C}} dz \frac{f(z)}{(1+z)^2} = 0. \quad (37)$$

where \mathcal{C} is the unit circle $|z| = 1$. Choosing $\kappa = 1$, the solution of (35) can be easily be found

$$f(z) = Az^{\frac{\alpha-1}{2}} \left(\frac{z}{\alpha+1} + \frac{1}{\alpha-1} \right) + B, \quad \alpha \neq 1, -1. \quad (38)$$

where A and B are integration constants. The values $\alpha = 1, -1$ are excluded because the associated $f(z)$ functions contain $\log z$ such that the condition (36) is not satisfied. From (36) and (38) one finds

$$f(e^{\pm i\pi}) = 0 \rightarrow B = \frac{2A}{1-\alpha^2} e^{\pm i\frac{\pi}{2}(\alpha-1)} \rightarrow e^{i\pi\alpha} = -1 \quad (39)$$

which yields the quantization conditions

$$\alpha_{n,\pm} = \begin{cases} (2n+3), & n = 0, 1, \dots, \quad \alpha > 0 \\ (2n+3), & n = -3, -4, \dots, \quad \alpha < 0 \end{cases} \quad (40)$$

that coincide with the semiclassical result (31) for $\kappa = 1$. The expression (38) for the positive energy solutions is

$$f_{\kappa=1,n,+}(z) = (-1)^n [(n+1)z^{n+2} + (n+2)z^{n+1} + (-1)^n] \quad (41)$$

and for the negative energy solutions

$$f_{\kappa=1,-n-3,-}(z) = (-1)^n \frac{n+1}{n} f_{\kappa=1,n,+}(z^{-1}), \quad n = 0, 1, \dots \quad (42)$$

The factor multiplying $f_{\kappa=1,n,+}(z^{-1})$ will be explained below. Finally, to verify the condition (37), we express (41) as

$$f_{\kappa=1,n,+}(z) = (z+1)^2 \sum_{r=0}^n (r+1)(-z)^r. \quad (43)$$

The vanishing of the integral follows from the Cauchy theorem. The solutions $f_{\kappa=1,n,-}(z)$ also satisfy eq.(37), which can be written as

$$\int_{\mathcal{C}} dz \frac{f(z^{-1})}{(1+z)^2} = 0. \quad (44)$$

Let us next consider generic values of κ . Eq.(35) has two linear independent solutions given by

$$\begin{aligned} f_{\kappa,n,+}(z) &= (z+1)^{\kappa+1} F(\kappa+1, -n, -\kappa-n+1, -z), \\ f_{\kappa,n,-}(z) &= (z+1)^{\kappa+1} z^{\kappa+n} F(\kappa, 2\kappa+n+1, \kappa+n+1, -z) \end{aligned} \quad (45)$$

where F is the hypergeometric function of type $F_{2,1}$, and α has been parametrized as

$$\alpha = 2n + 2\kappa + 1, \quad (46)$$

with n an arbitrary number. One can transform these eqs. into [27]

$$\begin{aligned} f_{\kappa,n,+}(z) &= (z+1)^{1-\kappa} F(-2\kappa-n, -\kappa+1, -\kappa-n+1, -z), \\ f_{\kappa,n,-}(z) &= (z+1)^{1-\kappa} z^{\kappa+n} F(n+1, -\kappa, \kappa+n+1, -z). \end{aligned} \quad (47)$$

The Gauss series defining these hypergeometric functions are absolutely convergent in the unit cycle $|z| = 1$ [27]. The condition (36) implies

$$\begin{aligned} F(-2\kappa-n, -\kappa+1, -\kappa-n+1, 1) &= \frac{\Gamma(2\kappa)\Gamma(-\kappa-n+1)}{\Gamma(\kappa+1)\Gamma(-n)} = 0 \\ F(n+1, -\kappa, \kappa+n+1, 1) &= \frac{\Gamma(2\kappa)\Gamma(\kappa+n+1)}{\Gamma(\kappa)\Gamma(2\kappa+n+1)} = 0 \end{aligned} \quad (48)$$

For generic values of κ , the first of these eqs. implies that $n = 0, 1, \dots$, and correspond to the positive energy solutions, while the second eq. implies $n = -(2\kappa + p + 1)$ ($p = 0, 1, \dots$), and correspond to the negative energy solutions $\alpha = -(2p + 2\kappa + 1)$. The relation between both solutions is given by

$$f_{\kappa,-n-2\kappa-1,-}(z) = (-1)^n \frac{\kappa+n}{\kappa} f_{\kappa,n,+}(z^{-1}), \quad n = 0, 1, \dots \quad (49)$$

and can be proved using hypergeometric identities [27]. The case $\kappa = 1$ reproduces eq.(42). As follows from (45), the functions $f_{\kappa,n,+}(z)$ are the product of a polynomial of degree n times the factor $(z+1)^{\kappa+1}$. Some examples are

$$\begin{aligned} f_{\kappa,0,+}(z) &= (z+1)^{\kappa+1} \\ f_{\kappa,1,+}(z) &= (z+1)^{\kappa+1} (1 - (1 + \kappa^{-1})z) \\ f_{\kappa,2,+}(z) &= (z+1)^{\kappa+1} (1 - 2z + (1 + 2\kappa^{-1})z^2) \\ &\dots \end{aligned} \quad (50)$$

which for $\kappa = 1$, reproduces eqs.(43). Finally, the condition (37) is also satisfied by the Cauchy theorem.

The wave functions $\psi(x)$ can be written as (recall eqs.(21) and (32))

$$\psi = (\cos \theta)^{1/2} \phi(\theta) = \frac{f(\theta)}{2(\cos \theta)^{1/2}}, \quad (51)$$

and the scalar product as

$$\begin{aligned}\langle \psi_1 | \psi_2 \rangle &= \int_{-\infty}^{\infty} dx \psi_1^*(x) \psi_2(x) = R \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \phi_1^*(\theta) \phi_2(\theta) \\ &= \frac{R}{4} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d\theta}{(\cos \theta)^2} f_1^*(\theta) f_2(\theta) = \frac{R}{2i} \int_C \frac{dz}{(z+1)^2} f_1^*(z) f_2(z).\end{aligned}\quad (52)$$

Using this formula one can compute the norm of the wave functions. For example for $\kappa = 1$ one gets

$$\langle \psi_{\kappa=1,n,+} | \psi_{\kappa=1,n,+} \rangle = \pi R(n+1)(n+2), \quad n = 0, 1, \dots \quad (53)$$

5. $SO(2,1)$ symmetry

The AdS_2 metric has the isometry group $SO(2,1)$. We then expect that this symmetry group can be realized in the xp - AdS_2 model. In this section we shall construct the generators of this group, first in the classical theory and then in the quantum theory. The group $SO(2,1)$ is isomorphic to $SU(1,1)$ whose generators, L_n ($n = 0, \pm 1$), satisfy the Poisson brackets

$$\{L_0, L_{\pm 1}\} = \pm i L_{\pm 1}, \quad \{L_1, L_{-1}\} = -2i L_0. \quad (54)$$

Choosing L_0 proportional to the classical Hamiltonian (1), i.e.

$$L_0 = \frac{R}{2w_0} H, \quad (55)$$

one finds

$$\begin{aligned}L_0 &= \frac{R}{2} \cosh\left(\frac{x}{R}\right) \left(p + \frac{\ell_p^2}{p}\right), \\ L_{\pm 1} &= \frac{R}{2} \cosh\left(\frac{x}{R}\right) \left(p e^{\mp 2i \tan^{-1}(\sinh(x/R))} - \frac{\ell_p^2}{p}\right).\end{aligned}\quad (56)$$

Upon quantization the algebra (54) becomes

$$[\hat{L}_0, \hat{L}_{\pm 1}] = \mp \hbar \hat{L}_{\pm 1}, \quad [\hat{L}_1, \hat{L}_{-1}] = 2\hbar \hat{L}_0, \quad (57)$$

and it is satisfied by the following normal ordered version of (56)

$$\begin{aligned}\hat{L}_0 &= \frac{R}{2} u(x) \left(\hat{p} + \frac{\ell_p^2}{\hat{p}}\right) u(x), \quad u(x) = (\cosh(x/R))^{1/2}, \\ \hat{L}_{\pm 1} &= \frac{R}{2} u(x) \left(e^{\mp i \tan^{-1}(\sinh(x/R))} \hat{p} e^{\mp i \tan^{-1}(\sinh(x/R))} - \frac{\ell_p^2}{\hat{p}}\right) u(x).\end{aligned}\quad (58)$$

To verify eqs.(57), it is convenient to perform the similarity transformation

$$\hat{A}' = u \hat{A} u^{-1}, \quad (59)$$

which amounts to act on the wave function ϕ related to ψ by eq.(21), i.e.

$$\hat{A}\psi = u^{-1}\hat{A}'\phi. \quad (60)$$

The transformation of \hat{L}_n under (59) is given in the variable θ by

$$\begin{aligned} (\hat{L}'_0\phi)(\theta) &= -\frac{i\hbar}{2}\partial_\theta\phi(\theta) - \frac{i\hbar\kappa^2}{2}\frac{1}{\cos\theta}\int_\theta^{\pi/2}\frac{d\theta'}{\cos\theta'}\phi(\theta'), \\ (\hat{L}'_{\pm 1}\phi)(\theta) &= -\frac{i\hbar}{2}e^{\mp 2i\theta}(\partial_\theta \mp i)\phi(\theta) + \frac{i\hbar\kappa^2}{2}\frac{1}{\cos\theta}\int_\theta^{\pi/2}\frac{d\theta'}{\cos\theta'}\phi(\theta'). \end{aligned} \quad (61)$$

The eigenfunctions of the Hamiltonian \hat{H} imply

$$\hat{L}'_0\phi_{\kappa,n,\pm} = \pm\hbar(n + \kappa + \frac{1}{2})\phi_{\kappa,n,\pm}, \quad n = 0, 1, \dots \quad (62)$$

Moreover, the positive and negative energy solutions satisfy

$$\hat{L}'_{\pm 1}\phi_{\kappa,0,\pm} = 0. \quad (63)$$

so that the positive energy eigenfunctions can be obtained acting on $\phi_{\kappa,0,+}$ with the raising operator $(\hat{L}'_{-1})^n$. Similarly, the negative energy eigenfunctions can be constructed acting on $\phi_{\kappa,0,-}$ with $(\hat{L}'_1)^n$. These two infinite dimensional representations of $SO(2,1)$ are related by complex conjugation.

6. The xp - AdS_2 model and conformal quantum mechanics

The group $SO(2,1)$ describes the symmetry of conformal quantum mechanics (CFT_1). The most studied model of a CFT_1 was introduced by Jackiw in 1972 [15], and its properties were analyzed in great detail by de Alfaro, Fubini and Furlan (dAFF) [16]. The dAFF model has been recently discussed in the framework of the AdS_2/CFT_1 correspondence [20, 21, 19]. In this section we shall discuss the relation of these works with the xp - AdS_2 model, which in turn may shed new light into the AdS_2/CFT_1 correspondence.

The dAFF Hamiltonian describes a particle on a half-line subject to an inverse square interaction potential, i.e.

$$H = \frac{1}{2}(p^2 + \frac{g}{x^2}), \quad x > 0, \quad g > 0, \quad (64)$$

which together with the operators D and K , defined as

$$\begin{aligned} D &= tH - \frac{1}{4}(xp + px), \\ K &= -t^2H + 2tD + \frac{1}{2}x^2, \end{aligned} \quad (65)$$

generate the $SO(2,1)$ algebra [16, 20]

$$i[D, H] = H, \quad i[D, K] = -K, \quad i[K, H] = 2D. \quad (66)$$

The Hamiltonian H does not have a discrete spectrum. However, one can construct a set of operators generating the $SO(2, 1)$ algebra in the Cartan basis (57) [16, 20]

$$\begin{aligned} L_0 &= \frac{1}{2} \left(\frac{K}{a} + aH \right), \\ L_{\pm 1} &= \frac{1}{2} \left(\frac{K}{a} - aH \right) \mp iD, \end{aligned} \quad (67)$$

where $\hbar a$ has dimensions of (length)², and such that L_0 has a discrete spectrum, i.e.

$$\begin{aligned} L_0 |n\rangle &= \hbar r_n |n\rangle, \\ r_n &= n + r_0, \quad n = 0, 1, \dots, \quad r_0 > 0, \\ \langle n | n' \rangle &= \delta_{n, n'}. \end{aligned} \quad (68)$$

The action of the operators $L_{\pm 1}$ acting in this basis is

$$L_{\pm 1} |n\rangle = \hbar \sqrt{r_n(r_n \mp 1) - r_0(r_0 - 1)} |n \mp 1\rangle. \quad (69)$$

The parameter r_0 labels the representation of the algebra and is related to the Casimir \mathbf{L}^2 of $SO(2, 1)$ as

$$\mathbf{L}^2 |n\rangle = \hbar^2 r_0(r_0 - 1) |n\rangle, \quad \mathbf{L}^2 = L_0^2 - \frac{1}{2}(L_1 L_{-1} + L_{-1} L_1). \quad (70)$$

In the dAFF model, r_0 is given by

$$r_0 = \frac{1}{2} \left(1 + \sqrt{g + \frac{1}{4}} \right). \quad (71)$$

On the other hand, the value of r_0 in the xp - AdS_2 model is given by (see (62))

$$r_0 = \kappa + \frac{1}{2}. \quad (72)$$

Although the dAFF and the xp - AdS_2 are two different models, one can establish a correspondence between their parameters based on dimensional arguments and their spectrum

$$\sqrt{g} \leftrightarrow \frac{R\ell_p}{\hbar}, \quad a \leftrightarrow \frac{R^2}{\hbar}, \quad (73)$$

which links a to the radius of the AdS_2 space, and g to that radius measured in units of the length \hbar/ℓ_p . An appealing feature of the xp - AdS_2 model is that the Hamiltonian is essentially the generator L_0 , unlike the dAFF model where L_0 involves a combination of the H and K operators. Therefore, it seems that contrarily to the time variable in the dAFF Hamiltonian, the time coordinate of the xp - AdS_2 is a suitable global time coordinate on AdS_2 . This result suggests that xp - AdS_2 model offers a more natural framework to study the AdS_2/CFT_1 correspondence [28].

7. Conclusions

We have shown in this paper that the generalized xp Hamiltonian defined on the AdS_2 spacetime has a harmonic oscillator spectrum with a zero point energy that depends on the radius of spacetime measured in units of a fundamental length in the model. The underlying symmetry group $SO(2, 1)$ of the AdS_2 metric has been realized in terms of operators of the quantum mechanical model. We have thus found an interesting model of conformal quantum mechanics which may shed new light into the AdS_2/CFT_1 correspondence.

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Appendix A. Short review on the AdS_2 spacetime

The AdS_2 spacetime is the locus of the hyperboloid [29]

$$X_0^2 + X_2^2 - X_1^2 = R^2, \quad (A.1)$$

where R is a length denoted the radius of AdS. The spacetime metric is inherited from the ambient Minkowski spacetime,

$$(ds)^2 = -dX_0^2 - dX_2^2 + dX_1^2, \quad (A.2)$$

which shows that $SO(2, 1)$ is the isometry group, whose compact subgroup $SO(2)$ can be identified with time. The hyperboloid (A.1) can be described in global coordinates as

$$\begin{aligned} X_0 &= R \cosh \rho \cos \tau, \\ X_2 &= R \cosh \rho \sin \tau, \\ X_1 &= R \sinh \rho, \end{aligned} \quad (A.3)$$

where $\rho \in (-\infty, \infty)$ and $\tau \in [0, 2\pi)$. The universal covering of AdS_2 spacetime is obtained letting τ to take any real value. The metric in these coordinates is

$$ds^2 = R^2(-\cosh^2 \rho d\tau^2 + d\rho^2). \quad (A.4)$$

The boundary at infinity, $\rho = \pm\infty$, consists of two disconnected worldlines parameterized by the time coordinate $\tau \in (-\infty, \infty)$. The coordinate ρ is identified with x/R in the xp - AdS_2 model. Throughout this paper we have used the variable θ defined as

$$\sinh \rho = \tan \theta, \quad \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right). \quad (A.5)$$

The boundaries of the spacetime are at $\theta = \pm\pi/2$. This coordinate displays clearly the causal structure of the metric (A.4), i.e.

$$ds^2 = \frac{R^2}{\cos^2 \theta} (-d\tau^2 + d\theta^2). \quad (\text{A.6})$$

Appendix B. Classical trajectories and geodesics

The equations of motion associated to the Hamiltonian (1) are [11]

$$\begin{aligned} \dot{x} &= w(x) \left(1 - \frac{\ell_p^2}{p^2}\right), \\ \dot{p} &= -w'(x) \left(p + \frac{\ell_p^2}{p}\right). \end{aligned} \quad (\text{B.1})$$

The energy E is a conserved quantity,

$$E = w(x) \left(p + \frac{\ell_p^2}{p}\right), \quad (\text{B.2})$$

as well as the sign of the momenta p , which coincides with the sign of E since $w(x) > 0$, $\forall x$. We shall assume below that $E, p > 0$. For each position x there are two possible values of the momenta,

$$p_\eta(x, E) = \frac{1}{2w(x)} (E + \eta \sqrt{E^2 - (2\ell_p w(x))^2}), \quad \eta = \pm 1, \quad (\text{B.3})$$

related by

$$p_-(x, E) = \frac{\ell_p^2}{p_+(x, E)}. \quad (\text{B.4})$$

The classical allowed region is given by

$$E \geq 2\ell_p w(x). \quad (\text{B.5})$$

Replacing (B.3) into (B.1) yields

$$\dot{x} = \frac{1}{2\ell_p^2 w(x)} \left((2\ell_p w(x))^2 - E^2 + \eta E \sqrt{E^2 - (2\ell_p w(x))^2} \right), \quad (\text{B.6})$$

so that the classical trajectories are given by

$$\int_{x_0}^x dx \frac{2\ell_p^2 w(x)}{(2\ell_p w(x))^2 - E^2 + \eta E \sqrt{E^2 - (2\ell_p w(x))^2}} = t - t_0. \quad (\text{B.7})$$

To integrate this eq. we use the variable θ defined in (7) or (A.5) and

$$w(x) = w_0 \cosh \frac{x}{R} = \frac{w_0}{\cos \theta}. \quad (\text{B.8})$$

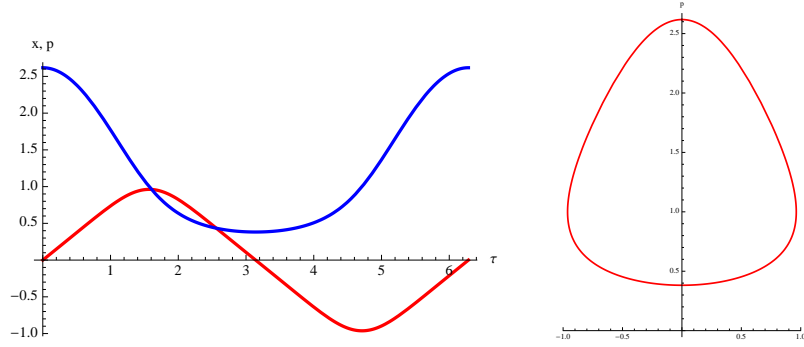


Fig.1-Left: Position (red line) and momentum (blue line) of a classical trajectory with $E = 3$ in units $w_0 = \ell_p = R = 1$. Right: Same trajectory in phase-space.

After some algebra (B.7) becomes (we choose $x_0 = t_0$ so that $\theta_0 = 0$)

$$\frac{R}{2w_0} \int_0^\theta d\theta \left[1 + \frac{\eta \varepsilon \cos \theta}{\sqrt{\varepsilon^2 \cos^2 \theta - 1}} \right] = t, \quad (\text{B.9})$$

where we have defined (recall (11))

$$\varepsilon = \frac{E}{2w_0 \ell_p} \geq 1. \quad (\text{B.10})$$

Performing the integral one finds

$$\theta + \eta \tan^{-1} \left[\frac{\varepsilon \sin \theta}{\sqrt{\varepsilon^2 \cos^2 \theta - 1}} \right] = \frac{2w_0 t}{R} = \tau + \theta. \quad (\text{B.11})$$

Hence, the classical trajectories in the global coordinates are

$$\tau = \eta \tan^{-1} \left[\frac{\varepsilon \sin \theta}{\sqrt{\varepsilon^2 \cos^2 \theta - 1}} \right], \quad (\text{B.12})$$

whose inverse

$$\theta(\tau) = \text{sign}(\pi - \tau) \cos^{-1} \sqrt{\cos^2 \tau + \frac{1}{\varepsilon^2} \sin^2 \tau}, \quad 0 \leq \tau \leq 2\pi, \quad (\text{B.13})$$

shows that they are periodic with the same period 2π for all energies. Finally, the position and momenta are given by

$$\begin{aligned} x(\tau) &= \sinh^{-1}(\tan \theta(\tau)), \quad 0 \leq \tau \leq 2\pi, \\ p(\tau) &= \ell_p(\varepsilon \cos \theta(\tau) + \sqrt{\varepsilon^2 - 1} \cos \tau). \end{aligned} \quad (\text{B.14})$$

Fig. 1 shows an example of a classical trajectory. The phase space contour is traversed in counterclockwise so that the Maslov phase is -2π , as for the standard harmonic oscillator. This justifies the constant $\frac{1}{2}$ in the semiclassical formula (9) for the energy levels.

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